Lecture 13 on Oct. 28 2013

Since today, we starting learning integrations of complex functions on curves. First of all, we make some assumptions on the curves in the remaining study.

Definition 0.1 (parametrization)**.** *Given a curve γ, we call $z(t) = x(t) + iy(t)$ a parametrization of γ if $z(t)$ is a one-one correspondence between some interval $[a, b]$ and points on γ.*

For our current purpose, we require $z(t)$ is piecewisely derivable and meanwhile $z(t)$ is one-one, which rule out the self-intersection. Noticing that in some cases, $z(a) = z(b)$ might happen. In this case, γ is called a closed curve. As usual, we call $z(a)$ the initial point on γ and $z(b)$ the ending point on γ if a parametrization $z(t)$ is given. By initial and end points, we can classify all parametrizations into two classes. In fact, letting A and B be two end points of γ, then we define

$$\mathscr{A}_1 = \{\text{All parametrizations with initial point } A \text{ and end point } B\ \}.$$

Moreover, we can also define

$$\mathscr{A}_2 = \{\text{All parametrizations with initial point } B \text{ and end point } A\ \}.$$

Now we pick up two parametrization from $\mathscr{A}_1$, say $z_1(t)$ and $z_2(\tau)$. Given t, we can find out $z_1(t)$ on γ by the first parametrization. Since the second parametrization is one-one, we can all find a $\tau(t)$ such that $z_2(\tau(t)) = z_1(t)$. Therefore we construct a one-one correspondence $\tau(t)$ between t-variable and τ-variable.

Proposition 0.2. *$\tau(t)$ is a one-one function.*

Proof. if $t \neq s$, then $z_1(t) \neq z_1(s)$. Hence $z_2(\tau(t)) \neq z_2(\tau(s))$. Moreover $\tau(t) \neq \tau(s)$. □

By Proposition 0.2, one knows that either τ is monotone increasing or decreasing. Since we assume z_1 and z_2 are from $\mathscr{A}_1$, therefore A is the initial point of both z_1 and z_2. Assuming that z_1 is defined on $[a, b]$ and z_2 is defined on $[c, d]$. Then we know that $z_1(a) = z_2(c) = A$. Samely, B is the end point for both z_1 and z_2, therefore we have $z_1(b) = z_2(d) = B$. From the above arguments, we know that $\tau(a) = c$ and $\tau(b) = d$. In light that $a < b$ and $c < d$, we can easily show that τ is increasing. One can also show that if z_1 and z_2 are from different classes, then the similarly define $\tau(t)$ must be decreasing. Hence we conclude that

Proposition 0.3. *If $z_1(t)$ and $z_2(t)$ are from same classes, then there is $\tau(t)$ monotone increasing such that $z_1(t) = z_2(\tau(t))$. If z_1 and z_2 are from different classes, then there is $\tau(t)$ monotone decreasing such that $z_1(t) = z_2(\tau(t))$.*

Now we introduce integrations on complex functions.

Definition 0.4. *If $u(t)$ and $v(t)$ are two real-valued functions, then we define*

$$\int_a^b u(t) + iv(t)\,\mathrm{d}t = \int_a^b u(t)\,\mathrm{d}t + i\int_a^b v(t)\,\mathrm{d}t.$$

Definition 0.5. *Given a curve γ and a parametrization $z(t)$, then we define*

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(z(t))z'(t)\,\mathrm{d}t.$$

Similarly, we also define

$$\int_\gamma f(z)\,\mathrm{d}\overline{z} = \int_a^b f(z(t))\overline{z'(t)}\,\mathrm{d}t.$$

Definition 0.6. *With Definition 0.5, we define the first-type line integral as follows:*

$$\int_\gamma f(z)\,\mathrm{d}x = \int_\gamma f(z)\,\frac{\mathrm{d}z + \mathrm{d}\overline{z}}{2} = \frac{1}{2}\int_\gamma f(z)\,\mathrm{d}z + \frac{1}{2}\int_\gamma f(z)\,\mathrm{d}\overline{z} = \int_a^b f(z(t))x'(t)\,\mathrm{d}t.$$

Samely we define

$$\int_\gamma f(z)\,\mathrm{d}y = \int_\gamma f(z)\,\frac{\mathrm{d}z - \mathrm{d}\overline{z}}{2i} = \frac{1}{2i}\int_\gamma f(z)\,\mathrm{d}z - \frac{1}{2i}\int_\gamma f(z)\,\mathrm{d}\overline{z} = \int_a^b f(z(t))y'(t)\,\mathrm{d}t.$$

In our following study, we also require the second-type line integral.

Definition 0.7.

$$\int_\gamma f(z)\,\mathrm{d}s = \int_\gamma f(z)\,|\mathrm{d}z| = \int_a^b f(z(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,\mathrm{d}t.$$

With the above definitions, some trivial properties can be easily obtained.

Proposition 0.8. *(1). for any complex number c, we have*

$$\int_a^b cf(t)\,\mathrm{d}t = c\int_a^b f(t)\,\mathrm{d}t.$$

(2) for any integrable complex function $f(t)$, we have

$$\mathrm{Re}\int_a^b f(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}f(t)\,\mathrm{d}t, \qquad \mathrm{Im}\int_a^b f(t)\,\mathrm{d}t = \int_a^b \mathrm{Im}f(t)\,\mathrm{d}t$$

(3) Moreover we have

$$\left|\int_a^b f(t)\,\mathrm{d}t\right| \le \int_a^b |f(t)|\,\mathrm{d}t.$$

(4) The above estimate also shows that

$$\left|\int_\gamma f(z)\,\mathrm{d}z\right| \le \int_\gamma |f(z)|\,|\mathrm{d}z|.$$

we now take a closer look at the integral in Definition 0.5. Firstly the integral must depend on parametrization $z(t)$. Supposing we have $z_1(t)$ and $z_2(\tau)$, two parametrizations of γ, then we can find $\tau(t)$ such that $z_1(t) = z_2(\tau(t))$. By definition 0.5, it holds

$$\int_\gamma f(z_1)\,\mathrm{d}z_1 = \int_a^b f(z_1(t))z_1'(t)\,\mathrm{d}t, \qquad \int_\gamma f(z_2)\,\mathrm{d}z_2 = \int_c^d f(z_1(\tau))z_2'(\tau)\,\mathrm{d}\tau.$$

Here $a < b$ and $c < d$. By change of variable, we know that

$$\int_\gamma f(z_2)\,\mathrm{d}z_2 = \int_c^d f(z_2(\tau))z_2'(\tau)\,\mathrm{d}\tau = \int_{\tau^{-1}(c)}^{\tau^{-1}(d)} f(z_2(\tau(t)))z_2'(\tau(t))\tau'(t)\,\mathrm{d}t.$$

Using the chain rule and the fact that $z_1(t) = z_2(\tau(t))$, the above equality can be reduced to

$$\int_\gamma f(z_2)\,\mathrm{d}z_2 = \int_{\tau^{-1}(c)}^{\tau^{-1}(d)} f(z_1(t))z_1'(t)\,\mathrm{d}t.$$

If z_1 and z_2 are from same class, say $\mathscr{A}_1$, then $\tau(t)$ is increasing. Therefore $\tau^{-1}(c) = a$ and $\tau^{-1}(d) = b$. This shows that

$$\int_\gamma f(z_2)\,\mathrm{d}z_2 = \int_a^b f(z_1(t))z_1'(t)\,\mathrm{d}t = \int_\gamma f(z_1)\,\mathrm{d}z_1.$$

If z_1 and z_2 are from different classes, then τ is decreasing. Therefore $\tau^{-1}(c) = b$ and $\tau^{-1}(d) = a$. This shows that

$$\int_\gamma f(z_2)\,\mathrm{d}z_2 = \int_b^a f(z_1(t))z_1'(t)\,\mathrm{d}t = -\int_\gamma f(z_1)\,\mathrm{d}z_1.$$

The above arguments show us that

Proposition 0.9. *Suppose that z_1 and z_2 are two parametrizations of γ. If z_1 and z_2 are from same class, then it holds*

$$\int_\gamma f(z_1)\,\mathrm{d}z_1 = \int_\gamma f(z_2)\,\mathrm{d}z_2.$$

Otherwise,

$$\int_\gamma f(z_1)\,\mathrm{d}z_1 = -\int_\gamma f(z_2)\,\mathrm{d}z_2$$

Using the same arguments above, one can also show that

Proposition 0.10. *The integral in Definition 0.7 is independent of parametrization. Equivalently for any z_1 and z_2 two parametrizations of γ, we have*

$$\int_\gamma f(z_1)\,|\mathrm{d}z_1| = \int_\gamma f(z_1)\,|\mathrm{d}z_2|.$$

Finally we conclude this lecture with some examples

Example 1. Given γ the upper half of the unit circle, then

$$\int_\gamma z\,\mathrm{d}z = 0.$$

Example 2. Given γ a curve in $\mathbb{C}$, then

$$\int_\gamma \mathrm{d}z = z_{\text{end}} - z_{\text{initial}}.$$

Example 3. Given γ the unit circle, then

$$\int_\gamma \frac{1}{z}\,\mathrm{d}z = 2\pi i.$$

Here we assume γ is winding around the origin counterclockwisely. If clockwisely, the answer should be $-2\pi i$.